Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 4, pp. 551–560, 1996

0021-8928/96 $24.00+0.00

PII: S0021-8928(96)00070-6

SYNTHESIS OF SUBOPTIMAL CONTROL OF STOCHASTIC SYSTEMS USING A PROGNOSING MODEL†

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(*Received* 10 *February* 1994)

The problem of controlling a stochastic system subject to the criterion that a generalized performance functional should be a minimum [1] is considered. A method is proposed for solving the problem, based on an optimal control algorithm with a prognosing model [1] and on the method of integral support curves for solving the Cauchy problem for ordinary differential equations [2–4]. The controls are synthesized in analytic form. Copyright © 1996 Elsevier Science Ltd.

1. GENERAL FORMULATION OF THE PROBLEM

There are two formulations of the problem of analytical design, due respectively to Letov–Kalman [5] and Krasovskii [1]. We will consider Krasovskii's formulation as it applies to the stochastic system

$$\dot{y}+f(y,t)=\varphi(y,t)u+\psi(y,t)n_1 \tag{1.1}$$

$$y(t,y_0)=y_0,\quad y\in Y\subset R^1,\quad t\in[t_0,t_M]$$

where the functions $f(y, t)$, $\varphi(y, t)$, $\psi(y, t)$ are such that the existence and uniqueness conditions for solutions of Eq. (1.1) are satisfied. In addition, $f(y, t)$ satisfies a Lipschitz condition in y with Lipschitz constant L, $n_1(t) \in R^1$ is white Gaussian noise with mathematical expectation $M\langle n_1\rangle = 0$ and variance $M\langle n_1(t^*)n_1(t^{**})\rangle = S_1(t^*)\delta(t^* - t^{**})$, $S_1(t)$ is the spectral density of the noise, $\delta(\cdot)$ is the delta function, $u(t) \in U \subset R^1$ is a control that minimizes the mathematical expectation of a given generalized performance functional [1]

$$M\langle J\rangle = M\langle V_3[y(t_M,y_0)]+\int_{t_0}^{t_M}Q[y(t,y_0),t]dt+\frac{1}{2}\int_{t_0}^{t_M}\frac{u^2+u_{0p}^2}{k^2}dt\rangle \tag{1.2}$$

$V_3(\cdot)$, $Q(\cdot)$ are given positive-definite functions and u_{op} is the optimal control for system (1.1).

We shall consider the use of the prognosing-model method [1] to seek optimal controls of the scalar object (1.1), i.e. controls that minimize the generalized performance functional (1.2). The results will be extended to the multidimensional case later.

Suppose the observation equation is

$$z=h(y,t)+n_2 \tag{1.3}$$

where the function $h(y, t)$ is assumed to be jointly continuous in both arguments, and $n_2(t) \in R^1$ is the interference of the observation, which is Gaussian white noise with mathematical expectation $M\langle n_2\rangle = 0$ and variance $M\langle n_2(t^*)n_2(t^{**})\rangle = S_2(t^*)\delta(t^* - t^{**})$.

According to the separation principle [1] for the observation (1.3), if the estimation of the state of system (1.1) is sufficiently accurate in the sense of minimum root mean square error, the controls

$$u(t,y_0)=-k^2\varphi(\bar{y},t)\partial V(\bar{y},t)/\partial\bar{y}|_{y=\bar{y}(t,y_0)} \tag{1.4}$$

will be optimal in the sense of minimizing the functional (1.2), where the function $V(y, t)$ satisfies the equation

†*Prikl. Mat. Mekh.* Vol. 60, No. 4, pp. 553–563, 1996.

$$\partial V/\partial t - f(y,t)\partial V/\partial y = -Q(y,t)$$

subject to the boundary condition ($V(y, t_M) = V_3(y)$).

According to the algorithm of optimal control with a prognosing model [1], the optimization interval $[t_0, t_M]$ is divided into sufficiently short cycles $\Delta t_{m+1} = t_{m+1} - t_m$ ($m = \overline{0, M-1}$), in such a way that for each fixed $t = t_m$ we have the equality

$$x_m = \bar{y}_m, \quad m = \overline{0, M-1} \tag{1.5}$$

which determines the initial data t_m, x_m necessary to solve the equation of free motion

$$\dot{x} + f(x,t) = 0, \quad x \in X \subset R^1, \quad t \in [t_m, t_M] \tag{1.6}$$

where $x(t, x_m) = \xi(t, t_m, x_m)$ is the solution of Eq. (1.6) corresponding to the initial data t_m, x_m and $\bar{y}_m$ is an estimate of the state of system (1.1), supplied to the control system according to the results of the observation at time $t = t_m$.

Taking (1.4) and (1.5) into consideration, we have the following estimate at the mth step

$$\bar{u}_{m+1} = -k^2\varphi(\bar{y}_m, t_m)\partial V(x,t)/\partial x\big|_{\substack{x=\bar{y}_m \\ t=t_m}}, \quad \bar{u}_{m+1} = \bar{u}(t, \bar{y}_m) \tag{1.7}$$

The function $V(x, t)$ is specified in the form [1]

$$V[x(t,\bar{y}_m),t] = V_3[x(t_M,\bar{y}_m)] + \int_t^{t_M} Q[x(t,\bar{y}_m),t]dt, \quad t \in [t_m, t_M] \tag{1.8}$$

As can be seen from (1.7), in order to determine an optimal control $\bar{u}_{m+1}$ at the mth step ($m = 0, 1, \ldots, m-1$) using a prognosing model, one must evaluate the partial derivative $\partial V/\partial x$ at the point $x = \bar{y}_m$, $t = t_m$. In practice this is done numerically, requiring a large number of "runs" of the model (1.6) during each cycle Δt_{m+1} [1].

To construct an approximate solution of the equation of free motion (1.6), one can use the method of integral support curves (ISC) of the solution of the Cauchy problem for ordinary differential equations [2–4]. The advantage of that method over the traditional approach is that it enables one to represent the initial equation as an array of data, from which an approximate solution of the equation may be obtained in analytic form with a given accuracy, i.e.

$$|x(t,x_m) - \tilde{x}(t,x_m)| \le \varepsilon_0, \quad x_m \in X, \quad t \in [t_m, t_M] \tag{1.9}$$

where $\tilde{x}(t,x_m)$ is an approximate solution of Eq. (1.6) constructed using the ISC method for initial data x_m, t_m and ε_0 is the maximum admissible error in the integration of Eq. (1.6). Obviously, when this method is used the expression for the suboptimal control (1.7) may be obtained in analytic form.

In accordance with the ISC method [2], besides the mesh t_m ($m = 0, 1, \ldots, M$), we also introduce a mesh $t_{(k)}$ ($k = 0, 1, \ldots, K$), with the stipulation $[t_0, t_M] \subseteq [t_{(0)}, t_{(K)}]$. In a special case, the meshpoints t_m and t_k may coincide. In addition, for the variable $x_{(0)} = x(t_{(0)})$ we introduce a mesh $x_{(0)i}$ ($i = 0, 1, \ldots, I$) which, by analogy with [3], enables us to consider the family of particular solutions $x(t, x_{(0)} = x_{(0)i})$ of Eq. (1.6) corresponding to a sequence of different initial data $x_{(0)i}$ ($i = 0, 1, \ldots, I$)

$$x(t, x_{(0)} = x_{(0)i}) = \xi(t, t_{(0)}, x_{(0)i}), \quad x(t_{(0)}, x_{(0)} = x_{(0)i}) = x_{(0)i} \quad (x_{(0)i} - x_{(0),i-1} = \Delta x_{(0)i}) \tag{1.10}$$

Following the analogy with [2], we shall call this family the family of ISCs determined for the set of meshpoints t_k ($k = 0, 1, \ldots, K$) and $x_{(0)i}$ ($i = 0, 1, \ldots, I$) by the sequence of fixed values

$$x(t_{(k)}, x_{(0)} = x_{(0)i}) = \xi(t_{(k)}, t_{(0)}, x_{(0)i}) = x_{(k)i} \tag{1.11}$$

We shall also assume that, as applied to (1.6), one can obtain data arrays $X^{(M)} = \{X_0^{(M)}, \ldots, X_I^{(M)}\}$ and $Y^{[N]} = \{Y_0^{[N]}, \ldots, Y_k^{[N]}$, where $X_i^{(M)} = \{x_{(k)i}^{(j)}, k = \overline{0, K}, j = \overline{0, M_k - 1}\}$ ($i = \overline{0, I}$), $Y_k^{[N]} = \{x_{(k)i}^{[l]}, i = \overline{0, L}, l = \overline{0, N_i - 1}\}$ ($k = \overline{0, K}$). Here

$$x_{(k)i}^{(j)} = \partial^j x(t, x_{(0)}) / \partial t^j \Big|_{\substack{t=t_{(k)} \\ x_{(0)}=x_{(0)i}}}, \quad x_{(k)i}^{[l]} = \partial^l x(t, x_{(0)}) / \partial x_{(0)}^l \Big|_{\substack{t=t_{(k)} \\ x_{(0)}=x_{(0)i}}}$$

Using the algorithm of optimal control with prognosing model (1.1)–(1.8) and the method of ISCs of the Cauchy problem [2–4], it is required to develop a new approach to the computation of approximate (suboptimal) controls $\bar{u}_{m+1}$ ($m = 0, 1, \ldots, M-1$); this approach should enable us to form an analytic expression for the control at the preliminary synthesis stage, and should guarantee that, if the integration meshpoints in the ISC method are chosen in the optimal way, then

$$|\bar{u}_{m+1} - \bar{\bar{u}}_{m+1}| \leq \varepsilon_1, \quad m = \overline{0, M-1} \tag{1.12}$$

where $\bar{u}_{m+1}$ is a control of system (1.1) that is exact (optimal) in the sense of minimizing the functional (1.2), and ε_1 is the maximal admissible error in computing the control inputs.

2. CONSTRUCTION OF A SOLUTION OF THE EQUATION OF FREE MOTION

Construction of a solution based on $x_{(k)i}^{(j)}$

We will represent the exact solution of the Cauchy problem for Eq. (1.6) in the domain $X \times [t_0, t_M]$, for initial data $x_{(0)}$ and $t_{(0)}$, in the form

$$x(t, x_{(0)}) = \xi(t, t_{(0)}, x_{(0)}), \quad x(t_{(0)}, x_{(0)}) = x_{(0)}, \quad x_{(0)} \in X_{(0)} \subset X \tag{2.1}$$

To construct an analytic solution of Eq. (1.6) for the problem in question, we shall assume that the quantities $x_{(k)i}^{(j)}$, which may be calculated by numerical–analytical operations, are known. Thus, the first and second derivatives with respect to time, $x_{(k)i}^{(1)}$ and $x_{(k)i}^{(2)}$, respectively, may be computed from the formulae

$$x_{(k)i}^{(1)} = f(x_{(k)i}, t_{(k)}), \quad x_{(k)i}^{(2)} = f^{(1)}(x, t)\Big|_{\substack{t=t_{(k)} \\ x=x_{(k)i}}} + f(x_{(k)i}, t_{(k)}) f^{[1]}(x, t)\Big|_{\substack{t=t_{(k)} \\ x=x_{(k)i}}}$$

To approximate a solution of Eq. (1.6), by analogy with [4], we interpolate a sample of numerical data $x_{(q)i}^{(p)}$ ($q = \overline{0, K}, p = \overline{0, M_q - 1}$) corresponding to the ith ISC ($i = 0, 1, \ldots, 1$)

$$\tilde{x}(t, x_{(0)i}, X_i^{(M)}) = \sum_{q,p} \alpha_{qi}^{(p)}(X_i^{(M)}) \gamma_{qp}(t) \tag{2.2}$$

where

$$\tilde{x}(t_{(k)}, x_{(0)i}, X_i^{(M)}) = \sum_{q,p} \alpha_{qi}^{(p)}(X_i^{(M)}) \gamma_{qp}^{(j)}(t_{(k)}) = x_{(q)i}^{(j)} \tag{2.3}$$

$$\gamma_{qp}^{(j)}(t_{(k)}) = d^j \gamma_{qp}(t) / dt^j \big|_{t=t_{(k)}}, \quad j = \overline{0, M_k - 1}, \quad k = \overline{0, K}$$

The coefficients $\alpha_{qi}^{(p)}(X_i^{(M)})$ are found by solving the system of linear algebraic equations (2.3).

Fixing $k = 0, 1, \ldots, K$ and $j = 0, 1, \ldots, M_{k-1}$, we associate with the meshpoints $x_{(0)i}$ ($i = 0, 1, \ldots, 1$) the sequence of values $\alpha_{ki}^{(j)}(X_i^{(M)})$ and interpolate

$$\alpha_k^{(j)}(X^{(M)}) = \sum_r \beta_{kr}^{(j)}(X^{(M)}) \mu_r(x_{(0)}) \tag{2.4}$$

(summation from $r = 0$ to $r = I$). Here

$$\sum_r \beta_{kr}^{(j)}(X^{(M)}) \mu_r(x_{(0)i}) = \alpha_{ki}^{(j)}(X_i^{(M)}) \tag{2.5}$$

The coefficients $3_{kr}^{(j)}(X^{(M)})$ are found by solving the system of equations (2.5).

Thus, by (2.2)–(2.5), the approximate solution $\tilde{x}(t, x_{(0)}, X^{(M)})$ can be represented in the form

$$\tilde{x}(t, x_{(0)}, X^{(M)}) = \sum_{q,p} \alpha_q^{(p)}(x_{(0)}, X^{(M)}) \gamma_{qp}(t) = \sum_{q,p,r} \beta_{qr}^{(p)}(X^{(M)}) \mu_r(x_{(0)}) \gamma_{qp}(t) \tag{2.6}$$

where

$$\sum_{q,p,r} \beta_{qr}^{(p)}(X^{(M)})\mu_r(x_{(0)i})\gamma_{qp}^{(j)}(t_{(k)}) = x_{(k)i}^{(j)}$$

If the construction of the ISCs (2.2) utilizes the Hermite interpolation formula, it is natural to put $\gamma_{qp}(t) = H_{qp}(t)$, $\alpha_{ki}^{(j)}(X_i^{(M)}) = x_{(q)i}^{(p)}$. Then expression (2.2) becomes

$$\tilde{x}(t, x_{(0)i}, X_i^{(M)}) = \sum_{q,p} x_{(q)i}^{(p)} H_{qp}(t) = \sum_{q,p,d} x_{(q)i}^{(p)} \frac{1}{q!}\frac{1}{p!}\left[\frac{(t-t_{(q)})^{M_q}}{\Omega_K(t)}\right]^{(d)} \frac{\Omega_K(t)}{(t-t_{(q)})^{M_q-p-d}} \tag{2.7}$$

$$\Omega_K(t) = (t-t_{(0)})^{M_0}(t-t_{(1)})^{M_1}\ldots(t-t_{(K)})^{M_K}$$

(summation over d from $d = 0$ to $d = M_q - p - 1$).

Using the Lagrange polynomial $L_r(x_{(0)})$ in interpolation (2.4), we obtain

$$\alpha_k^{(j)}(x_{(0)}, X^{(M)}) = \sum_r x_{(k)r}^{(j)} L_r(x_{(0)}) = \sum_r x_{(k)r}^{(j)} \frac{\omega_l(x_{(0)})}{(x_{(0)} - x_{(0)r})\omega_l^{(1)}(x_{(0)})|_{x_{(0)}=x_{(0)r}}} \tag{2.8}$$

$$\omega_l(x_{(0)}) = (x_{(0)} - x_{(0)0})(x_{(0)} - x_{(0)1})\ldots(x_{(0)} - x_{(0)l})$$

$$\omega_l^{(1)}(x_{(0)}) = d\omega_l(x_{(0)})/dx_{(0)}$$

Thus, taking (2.7) and (2.8) into consideration, we can write the solution (2.6) as

$$\tilde{x}(t, x_{(0)}, X^{(M)}) = \sum(x_{(0)}, t, X^{(M)}) \tag{2.9}$$

$$\sum(x, t, X^{(M)}) = \sum_{q,p,r} x_{(q)r}^{(p)} L_r(x) H_{qp}(t)$$

Construction of a solution based on $x_{(k)i}^{[l]}$

In accordance with the formulation of the problem, when constructing a solution of Eq. (1.7) based on the data $Y^{[N]}$ we shall assume that the quantities $x_{(k)i}^{[l]}$ are known; they may be calculated by a numerical–analytic approach based on the use of successive approximations. To do this, we differentiate the following obvious expression as many times as necessary with respect to $x_{(0)}$ (over the interval $[t_{(0)}, t]$)

$$x(t, x_{(0)}) = \int f(\tau, \xi(\tau, t_{(0)}, x_{(0)}))d\tau + x_{(0)}$$

For example, the first and second derivatives are evaluated by the formulae

$$x^{[1]}(t, x_{(0)}) = \int x^{[1]}(\tau, x_{(0)}) f^{[1]}(x, \tau)d\tau + 1$$

$$x^{[2]}(t, x_{(0)}) = \int \{[x^{[1]}(\tau, x_{(0)})]^2 f^{[2]}(x, \tau) + x^{[2]}(\tau, x_{(0)}) f^{[1]}(x, t)\}d\tau$$

Taking the data array (1.11) into consideration, the algorithm for computing $x_{(k)i}^{[1]}$ and $x_{(k)i}^{[2]}$ reduces to the following procedures

$$x_n^{[1]}(t, x_{(0)i}) = \int x_{n-1}^{[1]}(\tau, x_{(0)i}) f^{[1]}(\tau, x(\tau, x_{(0)i}))d\tau + 1, \quad n = \overline{1, Q_1} \tag{2.10}$$

$$x_{(k)i}^{[1]} = x_{Q_1}^{[1]}(t_{(k)}, x_{(0)i})$$

$$x_n^{[2]}(t, x_{(0)i}) = \int\{[x_{Q_1}^{[1]}(\tau, x_{(0)i})]^2 f^{[2]}(\tau, x(\tau, x_{(0)i})) +$$

$$+ x_{n-1}^{[2]}(\tau, x_{(0)i}) f^{[1]}(\tau, x(\tau, x_{(0)i}))\}d\tau, \quad n = \overline{1, Q_2} \tag{2.11}$$

$$x_{(k)i}^{[2]} = x_{Q_2}^{[2]}(t_{(k)}, x_{(0)i})$$

These formulae are modifications of the well-known method of successive approximations, applied to the case in which one has highly accurate data on the family of ISCs. Similarly, successive differentiation with respect to $x_{(0)}$ yields appropriate formulae for derivatives $x^{[l]}_{(k)i}$ of higher orders.

Another approach to the computation of $x^{[l]}_{(k)i}$ is based on using Lindelöf's theorem, according to which the algorithm for computing, say, $x^{[1]}_{(k)i}$ reduces to the rule

$$x^{[1]}_{(k)i}=\exp\{\int f^{[1]}(\tau,x(\tau,x_{(0)}))d\tau\} \tag{2.12}$$

Let us interpolate the sample of numerical data $x^{[q]}_{(k)p}$ ($p=\overline{0,I}$, $q=\overline{0,N_P-1}$), for a fixed $k=0, 1, \ldots, K$

$$\tilde{x}(t_{(k)},x_{(0)},Y^{[N]}_k)=\sum_{p,q}\alpha^{[q]}_{kp}(Y^{[N]}_k)\mu_{pq}(x_{(0)}) \tag{2.13}$$

in which case

$$\tilde{x}^{[l]}(t_{(k)},x_{(0)i},Y^{[N]}_k)=\sum_{p,q}\alpha^{[q]}_{kp}(Y^{[N]}_k)\mu^{[l]}_{pq}(x_{(0)i})=x^{[l]}_{(k)i} \tag{2.14}$$

where

$$\mu^{[l]}_{pq}(x_{(0)i})=d^l\mu_{pq}(x_{(0)})/d^lx_{(0)}|_{x_{(0)}=x_{(0)i}},\quad l=\overline{0,N_i-1}$$

The coefficients $\alpha^{[q]}_{kp}(Y^{[N]}_k)$ are found by solving the system of equations (2.14).

Fixing $i=0, 1, \ldots, I$ and $1=0, 1, \ldots, N_i-1$, we can associate with the meshpoints $t_{(k)}$ ($k=0, 1, \ldots, K$) the set of values $\alpha^{[l]}_{ki}(Y^{[N]}_k)$ and interpolate

$$\alpha^{[l]}_i(Y^{[N]})=\sum_r\beta^{[l]}_{ri}(Y^{[N]})\gamma_r(t) \tag{2.15}$$

The coefficients $\beta^{[l]}_{ri}(Y^{[N]})$ are found by solving the system of equations

$$\sum_r\beta^{[l]}_{ri}(Y^{[N]})\gamma_r(t_{(k)})=\alpha^{[l]}_{ki}(Y^{[N]}_k) \tag{2.16}$$

(summation over r from $r=0$ to $r=K$).

Thus, in view of (2.13)–(2.16), the approximate solution $\tilde{x}(t, x_{(0)}, Y^{[N]})$ may be written in the form

$$\tilde{x}(t,x_{(0)},Y^{[N]})=\sum_{q,p}\alpha^{[q]}_p(t,Y^{[N]})\mu_{pq}(x_{(0)})=\sum_{q,p,r}\beta^{[q]}_{rp}(Y^{[N]})\mu_{pq}(x_{(0)})\gamma_r(t) \tag{2.17}$$

where

$$\sum_{q,p,r}\beta^{[q]}_{rp}(Y^{[N]})\mu^{[l]}_{pq}(x_{(0)i})\gamma_r(t_{(k)})=x^{[l]}_{(k)i} \tag{2.18}$$

If one uses Hermite's formulae for the interpolation (2.13) and a Lagrange polynomial ($\mu_{pq}(x_{(0)})=H_{pq}(x_{(0)})$, $\alpha^{[q]}_{kp}=x^{[q]}_{(k)p}$, $\gamma_r(t)=L_r(t)$) for the interpolation (2.15), the approximate solution (2.17) takes the form

$$\tilde{x}(t,x_{(0)},Y^{[N]})=\Sigma(x_{(0)},t,Y^{[N]}) \tag{2.19}$$

$$\Sigma(x,t,Y^{[N]})=\sum_{q,p,r}x^{[q]}_{(r)p}H_{pq}(x)L_r(t)$$

3. SYNTHESIS OF SUBOPTIMAL CONTROL

Let us assume that the functions $V_3(x)$ and $Q(x,t)$ are given quadratic forms, as is frequently the case in practice [1, 5]: $V_3(x)=\upsilon x^2/2$, $Q(x,t)=q(t)x^2/2$.

In that case, using (1.8), we can write (integration over the interval $[t_m, t_M]$)

$$\left.\frac{\partial V}{\partial x}\right|_{t=t_m}=\upsilon\,x(t_M,\bar{y}_m)+\int q(t)x(t,\bar{y}_m)dt \tag{3.1}$$

where $x(t, \bar{y}_m) = \xi(t, t_m, \bar{y}_m)$ is the solution of Eq. (1.6) for initial data $t_m, \bar{y}_m$.

Taking the last relationship into consideration, as well as (1.7), we obtain the control

$$\bar{u}_{m+1} = -k^2\varphi(\bar{y}_m, t_m)\{v\, x(t_M, \bar{y}_m) + \int q(t)x(t, \bar{y}_m)dt\} \tag{3.2}$$

Replacing the exact solution $x(t, \bar{y}_m)$ in formula (3.2) by the approximate solution $\tilde{x}(t, v_m)$ as given by (2.9), we find that the required suboptimal control may be written as

$$\bar{\bar{u}}_{m+1} = -k^2\varphi(\bar{y}_m, t_m)\{v\sum(\bar{y}_m, t_M, X^{(M)}) + \int q(t)\sum(\bar{y}_m, t, X^{(M)})dt\} \tag{3.3}$$

When the approximate solution $\tilde{x}(t, \bar{y}_m, Y^{[N]})$ is used, the expression for the control differs from (3.3) only in the replacement of the term $\Sigma(\bar{y}_m, t, X^{[M]})$ by a term $\Sigma(\bar{y}_m, t, Y^{[M]})$ as given by (2.19).

Hence, based on the specific form of the equation of free motion (1.6), one can select either of the two solutions to the problem of the analytic design of control systems. In practical applications, priority should be given to that solution which, while guaranteeing the prescribed accuracy for the synthesis of suboptimal controls, requires a smaller amount of a priori data about the family of ISCs. In other words, the decision as to which construction to choose should be based on an examination of the inequalities

$$\sum_q M_q \lesseqgtr \sum_p N_p$$

Note that in order to construct suboptimal controls of type (3.3) one can use not only Lagrange and Hermite polynomials, but also other interpolation structures, which may be more convenient from the standpoint of computer implementation.

With the controls determined above, the induced motion of system (1.1) at the mth step ($m = 0, 1, \ldots, M-1$) will obey the law

$$\frac{d\bar{\bar{y}}}{dt} + f(\bar{\bar{y}}, t) = \varphi(\bar{\bar{y}}, t)\bar{\bar{u}}_{m+1}$$
$$\bar{\bar{y}}(t_m \bar{y}_m) = \bar{y}_m, \quad t \in [t_m, t_{m+1}]$$

4. SELECTION OF INTERPOLATION POINTS

Let us consider the solution of our problem as it applies to the case in which only the array of numbers (1.11) is used to construct an approximate solution $\tilde{x}(t, x_{(0)}) = \tilde{x}(t, x_{(0)}, X^{(0)} = \tilde{x}(t, x_{(0)}, Y^{[0]})$ of Eq. (1.6). It is obvious that the true integral curve $x(t, x_{(0)i})$ differs from the approximate curve $\tilde{x}(t, x_{(0)i})$ at $t = t_k$ by an amount $\Delta x_{(k)i} = \tilde{x}(t_{(k)}, x_{(0)i}) - x(t_{(k)}, x_{(0)i})$ $(i \notin \overline{0, I})$. In view of (1.9), let us assume that $\tilde{x}(t, x_{(0)})$ is an ε-approximation in the sense of the mismatch in the solution, i.e. $|\tilde{x}(t, x_{(0)}) - x(t_{(0)}) - x(t_0, x_{(0)})| \leq \varepsilon$, and when the function $\tilde{x}(t, x_{(0)})$ is substituted into (1.6) we obtain

$$\dot{\tilde{x}} + f(\tilde{x}, t) = \psi(t)$$

where the mismatch $\psi(t)$ satisfies the inequality

$$\max_t |\psi(t)| \leq \varepsilon$$

We then obtain the following estimate [2] for the two continuously differentiable solutions $x(t, x_{(0)})$ and $\tilde{x}(t, x_0)$

$$\rho[(x(t, x_{(0)}), \tilde{x}(t, x_{(0)})] = \varepsilon \exp[L(t - t_{(0)})] + \frac{\varepsilon}{L}\{\exp[L(t - t_{(0)})] - 1\} \leq$$
$$\leq \varepsilon\left\{\exp[L(t_{(K)} - t_{(0)}) + \frac{1}{L}[\exp[L(t_{(K)} - t_{(0)})] - 1]\right\}$$

It is well known that the error in two-dimensional interpolation is the remainder term

$$x(t,x_{(0)})-\tilde{x}(t,x_{(0)})=\frac{\omega_K(t)}{(K+1)!}\frac{\partial^{K+1}}{\partial t^{K+1}}\xi(t^*,t_{(0)},x_{(0)})+$$
$$+\frac{\omega_I(x_{(0)})}{(I+1)!}\frac{\partial^{I+1}}{\partial x_{(0)}^{I+1}}\xi(t,t_{(0)},x^*_{(0)})-\frac{\omega_K(t)}{(K+1)!}\frac{\omega_I(x_{(0)})}{(I+1)!}\frac{\partial^{K+I+2}}{\partial t^{K+1}\partial x_{(0)}^{I+1}}\xi(t^{**},t_{(0)},x^{**}_{(0)}) \tag{4.1}$$

where t^*, t^{**} and $x^*_{(0)}$, $x^{**}_{(0)}$ are certain characteristic values of the variables t and $x_{(0)}$.

In view of (4.1), the following inequality may be used when the family of ISCs is being selected

$$|x(t,x_{(0)}-\tilde{x}(t,x_{(0)})|\leqslant\frac{Q_{K+1}}{(K+1)!}\overset{*}{\omega}_{K,1}+\frac{G_{I+1}}{(I+1)!}\overset{*}{\omega}_{I,2}-\frac{D_{K+1,I+1}}{(K+1)!(I+1)!}\overset{*}{\omega}_{K,1}\overset{*}{\omega}_{I,2}\leqslant\varepsilon_0 \tag{4.2}$$

where

$$\overset{*}{\omega}_{K,1}=\max_t|\omega_K(t)|,\quad \overset{*}{\omega}_{I,2}=\max_{x_{(0)}}|\omega_I(x_{(0)})|$$

$$Q_{K+1}=\max_{t,x_{(0)}}\left|\frac{\partial^{K+1}}{\partial t^{K+1}}\xi(t,t_{(0)},x_{(0)})\right|$$

$$Q_{I+1}=\max_{t,x_{(0)}}\left|\frac{\partial^{I+1}}{\partial t_{(0)}^{I+1}}\xi(t,t_{(0)},x_{(0)})\right|$$

$$D_{K+1,I+1}=\max_{t,x_{(0)}}\left|\frac{\partial^{K+I+2}}{\partial t^{K+1}\partial x_{(0)}^{I+1}}\xi(t,t_{(0)},x_{(0)})\right|$$

Having specified the value of ε_0, one can numerically select numbers K and I such that Eq. (1.6) can be integrated to within the required accuracy. The mesh-sizes $\Delta x_{(0)i}$ and $\Delta t_{(k)}$ should be chosen so as to minimize the error estimate for the two-dimensional interpolation (4.1). The interpolation points in that case should coincide with the roots of a Chebyshev polynomial, i.e. to find upper bounds for the quantities $\omega_k(t)$ and $\omega_i(x_{(0)})$ one can use the inequalities

$$\begin{aligned}\max_t|\omega_K(t)|&\leqslant\frac{(t_{(K)}-t_{(0)})^{K+1}}{2^{2K+1}}\\ \max_{x_{(0)}}|\omega_I(x_{(0)})|&\leqslant\frac{(x_{(0)I}-x_{(0)0})^{I+1}}{2^{2I+1}}\end{aligned} \tag{4.3}$$

Accordingly, instead of (4.2) we can use the estimate

$$|x(t,x_{(0)})-\tilde{x}(t,x_{(0)})|\leqslant\frac{(t_{(K)}-t_{(0)})^{K+1}}{2^{2K+1}}\frac{Q_{K+1}}{(K+1)!}+\frac{(x_{(0)I}-x_{(0)0})^{I+1}}{2^{2I+1}}\frac{G_{I+1}}{(I+1)!}-$$
$$-\frac{(t_{(K)}-t_{(0)})^{K+1}}{2^{2K+1}}\frac{(x_{(0)I}-x_{(0)0})^{I+1}}{2^{2I+1}}\frac{D_{K+1,I+1}}{(K+1)!(I+1)!}\leqslant\varepsilon_0 \tag{4.4}$$

which corresponds to selecting the interpolation points for t and $x_{(0)}$ via the formulae

$$t_{(k)}=\frac{1}{2}\left[(t_{(K)}-t_{(0)})\cos\frac{2k+1}{2K+2}\pi+t_{(K)}+t_0\right],\quad k=\overline{0,K} \tag{4.5}$$

$$x_{(0)i} = \frac{1}{2}\left[(x_{(0)l} - x_{(0)0})\cos\frac{2i+1}{2l+2}\pi + x_{(0)l} + x_{(0)0}\right], \quad i = \overline{0,l}$$

Now, considering $\bar{u}_{m+1}$ and $\bar{\tilde{u}}_{m+1}$ as exact (optimal) and approximate (suboptimal) controls, respectively, we introduce a measure of the mismatch

$$|\bar{u}_{m+1} - \bar{\tilde{u}}_{m+1}| = k^2|\varphi(\bar{y}_m, t_m)|\left|\frac{\partial V}{\partial x} - \frac{\partial \tilde{V}}{\partial x}\right|_{\substack{x=\bar{y}_m \\ t=t_m}} \tag{4.6}$$

where

$$\frac{\partial \tilde{V}}{\partial x} = \nu\,\tilde{x}(t_M, \bar{y}_m) + \int q(t)\tilde{x}(t, \bar{y}_m)dt \tag{4.7}$$

Taking (4.6) and (4.7) into consideration and using the triangle inequality, we can write

$$|\bar{u}_{m+1} - \bar{\tilde{u}}_{m+1}| \leq k^2|\varphi(\bar{y}_m, t_m)|[\nu\,|x(t_M, \bar{y}_m) - \tilde{x}(t_M, \bar{y}_m)| + \int q(t)|x(t, \bar{y}_m) - \tilde{x}(t, \bar{y}_m)|dt] \tag{4.8}$$

By (4.4) and (4.8), we have the inequality

$$|\bar{u}_{m+1} - \bar{\tilde{u}}_{m+1}| \leq k^2\,\mathring{\varphi}[\nu\,\varepsilon_0 + \mathring{q}(t_M - t_m)\varepsilon_0] = \varepsilon_1 \tag{4.9}$$

where

$$\mathring{q} = \max_t q(t), \quad \mathring{\varphi} = \max_{t,y}|\varphi(y, t)|$$

Hence, with ε_1 given, we can compute

$$\varepsilon_0 = \varepsilon_1[k^2\,\mathring{\varphi}(\nu + \mathring{q})]^{-1} \tag{4.10}$$

which in the final analysis determines the required accuracy of integration for Eq. (1.6) by the ISC method, so as to guarantee the construction of a control of the necessary quality.

The approach proposed here is also applicable to multidimensional systems with vector controls, but in that case the computer must have sufficient memory capacity to store data arrays corresponding to the system of differential equations for free motion.

5. EXAMPLE

Let us consider the problem of regulating the angular velocities of a spaceship, on the basis of the example presented in [6]. For brevity, we shall assume that the state of the spaceship can be estimated to within satisfactory accuracy and that the separation theorem holds, so that the control problem is solvable in a deterministic formulation [1].

Suppose the motion of a space ship with a single axis of symmetry is described by the equations

$$\dot{y}_1 + Ay_2y_3 = u_1, \quad \dot{y}_2 - Ay_1y_2 = u_2, \quad \dot{y}_3 = u_3, \quad t \in [t_0, t_M] \tag{5.1}$$

where y_1, y_2 and y_3 are the angular velocities ($y_j(t_0) = y_{j,0}$), A is the reduced moment of inertia, $t_0 = 0$ [s], and $t_M = 10$ [s]. Here and below $j = 1, 2, 3$.

It is required to determine a control that will steer object (5.1) up to time $t = t_M$ from the state $y_j(t_0) = y_{j,0}$ to the state $y_j(t_M)$, which is optimal in the sense that it minimizes the functional

$$M(J) = M\left\langle \sum_j \nu_{jj}y_j^2(t_M) + \int \sum_j q_{jj}y_j^2(t)dt + \int \sum_j \frac{u_j^2 + u_{iop}^2}{k_j^2}dt \right\rangle \tag{5.2}$$

where

$$V_3(y) = \sum_j \nu_{jj}y_j^2(t_M), \quad Q(y, t) = \sum_j q_{jj}y_j^2(t)$$

and v_{ij} and q_{ij} are given coefficients.

The system of equations of free motion, applied to (5.1), is, by analogy with (1.6),

$$\dot{x}_1 + Ax_2x_3 = 0, \quad \dot{x}_2 - Ax_1x_2 = 0, \quad \dot{x}_3 = 0 \tag{5.3}$$

Stipulating that the maximum admissible computation error in the controls is $\varepsilon_{1j} = 0.5$ [%], we use (4.10) to evaluate the accuracy with which system (5.3) must be integrated by the ISC method

$$\varepsilon_{0,j} = \varepsilon_{1,j}(v_{jj} + q_{jj})^{-1} \tag{5.4}$$

By (5.4) we have $\varepsilon_{0,1} = 0.3$ [%], $\varepsilon_{0,2} = 0.1$ [%], $\varepsilon_{0,3} = 0.1$ [%]. We also specify the domain in which the initial data may vary as

$$b_1 \le x_{1,0} \le d_1, \quad b_2 \le x_{2,0} \le d_2, \quad b_3 \le x_{3,0} \le d_3 \tag{5.5}$$

where

$$b_1 = 0 \ [\mathrm{c}^{-1}], \quad d_1 = 2{,}6 \ [\mathrm{c}^{-1}], \quad b_2 = -2 \ [\mathrm{c}^{-1}]$$

$$d_2 = 0 \ [\mathrm{c}^{-1}], \quad b_3 = 0 \ [\mathrm{c}^{-1}], \quad d_3 = 1{,}4 \ [\mathrm{c}^{-1}]$$

Putting $t_{(0)} = t_0$, $t_{(K)} = t_M$ and proceeding by analogy with (4.5), we select mesh-sizes $\Delta x_{j,(0)i} = x_{j,(0)i} - x_{j,(0)i-1}$ and $\Delta t_{(k)} = t_{(k)} - t_{(k-1)}$ so that the interpolation points are the roots of a Chebyshev polynomial

$$x_{j,(0)i} = \frac{d_j + b_j}{2} - \frac{d_j - b_j}{2}\cos\left(\frac{2i-1}{2I}\pi\right), \quad i = \overline{0,I} \tag{5.6}$$

$$t_{(k)} = \frac{t_{(K)} + t_{(0)}}{2} - \frac{t_{(K)} - t_{(0)}}{2}\cos\left(\frac{2k-1}{2K}\pi\right), \quad k = \overline{0,K} \tag{5.7}$$

In accordance with (4.4) and the values of ε_{0j}, we choose K and I to ensure the required computation accuracy. The results of the computations imply $I = 2$, $K = 3$. Using (5.6) and (5.7) to compute the optimal interpolation points, we construct a family of 54 ISCs using a fourth-order Runge–Kutta method. Using these ISCs and (5.3), we obtain the values $x^{(1)}_{1(k)i} = -Ax_{2,(k)i}x_{3,(k)i}$, $x^{(1)}_{2(k)i} = Ax_{1,(k)i}x_{2,(k)i}$, $x^{(1)}_{3(k)i} = 0$.

We have thus obtained an array of numbers $x^{(p)}_{j,(k)i}$ ($k = 0, 1, 2, 3$; $i = 0, 1, 2$; $p = 1, 2$), which enables us, using (2.7)–(2.9), to express the approximate solution of system (5.3) in the form

$$\tilde{x}_j(t, x_{(0)}, X^{(M)}) = \sum_{q=0}^{3}\sum_{p=0}^{1}\sum_{r=0}^{2} x^{(p)}_{j,(q)r} L_r(x_{j,(0)}) H_{qp}(t) \tag{5.8}$$

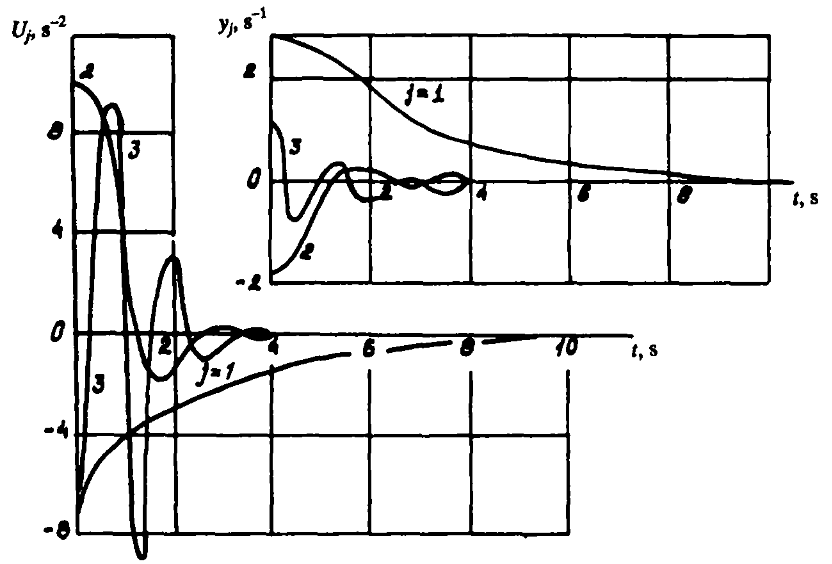

Fig. 1.

In view of this formula, the expression for the required suboptimal control will be

$$\bar{\bar{u}}_{j,m+1} = -k_j^2 \left\{ 2\sum_{l=1}^{3} \upsilon_{ll}\tilde{x}_l(t_M, \bar{y}_m, X^{(M)}) + 2\int \sum_{l=1}^{3} q_{ll}\tilde{x}_l(t, \bar{y}_m, X^{(M)})dt \right\} \tag{5.9}$$

We have thus carried out the computational procedures relating to the stage of preliminary synthesis and an analytic expression has been obtained for the suboptimal control. Specifying the initial data $y_{1,0} = 2.5$ [s^{-1}]; $y_{2,0} = -1.7$ [s^{-1}]; $y_{3,0} = 1.2$ [s^{-1}], $\upsilon_{11} = q_{11} = 0.9493$; $\upsilon_{22} = q_{22} = 2.3$; $\upsilon_{33} = q_{33} = 2.1$; $k_j = 1$, let us calculate a suboptimal control for $\Delta t_m = 0.5$ [s] and the corresponding phase trajectory of system (5.1). For a comparative analysis of the results, we also determined a control and trajectory for the same initial data, using the traditional algorithm with prognostic model. Analysis of the results showed that the relative error in computing the controls using the method proposed above was at most 10^{-4}, owing to the closeness of the approximate solution (5.8) to the solution obtained numerically.

Figure 1 shows graphs of the suboptimal control computed using (5.9) and the corresponding phase trajectory of the system (5.1).

6. CONCLUSION

For any optimal control algorithm with a prognosis model based on difference stencils [1], the model must be "run" at each step for the construction and numerical differentiation of $V(x, t)$ as a function of x. The number of such "runs" in the simplest two-point scheme to compute the derivative for a multidimensional object is $S + 1$, where S is the dimension of the control vector [1]. In an r-point difference scheme the number of "runs" reaches $(r - 1)S + 1$ [1], a figure that, in practice, may require enormous computational resources.

The use of the method developed in this paper enables one to circumvent the need for multiple "runs" of the kind described, since the solution of the equation of free motion, and hence also the required controls, are found in analytic form, and their computation requires only the substitution into formula (3.3) of the initial data supplied to the control system at each step.

Thus, the method is suitable for extensive use in problems of the analysis and synthesis of suboptimal systems for controlling objects described by non-linear differential equations—a fairly typical situation.

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Translated by D.L.